Patterns of Misperceptions in Linear Transformations: Four Illustrations

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Student misperceptions (visual unawareness) constitute a powerful hindrance to successful mathematical problem solving. Using the study's computer software, a number of lower secondary, upper secondary mathematics students, their teachers and other pre-service teachers were required to perform a number of transformations. These exercises demonstrated the extent to which misperceptions confound the process of conceptualisation in problem solving activity and mathematical learning. This presentation highlights the misperception phenomenon and describes remediation processes that are being developed in an effort to counteract the negative impact of the phenomenon.

Background

At the previous MERGA conference in Sydney, the authors described an ongoing study of misperceptions in music and mathematics (Malone, Leong & Lamb, 2001). The term 'misperception' was defined as the act of perceiving via a single sensory modality (e.g. seeing in maths or hearing in music) something that is different from reality or an imagined reality (e.g. visualising a reflected object/shape in maths, or auditing a transposed sound pattern in music). The presentation this year focuses on one small component of the larger study – namely, a demonstration that misperceptions do indeed exist in mathematics, and the production of evidence that these misconceptions can be overcome through training in the use of software developed specifically for the purpose. The mathematics topic selected to be the vehicle for this demonstration was one involving linear transformations. Brief explanations of the status of this topic in the Western Australian mathematics curriculum and the reasons for its choice in this study are appropriate at this point.

The Space Strand of the *WA Curriculum Framework* constitutes one of seven mathematics learning areas. The strand descriptor reads: 'Students will describe and analyse mathematically the spatial features of objects, environments and movements. In particular, they should visualise, draw and model shapes, locations and arrangements and reason about ways to solve related problems and justify solutions'. The strand is divided into four sub-strands, one of which is entitled 'Representing Transformations'. This sub-strand's objective reads: 'Students should visualise and show the effect of transformations on shapes and arrangements'.

Eight levels of outcome statements are associated with the Representing Transformations (RT) sub-strand, these levels representing standards of output considered achievable by students ranging from the lower and upper primary levels to the lower and upper secondary levels of schooling. For example, the Level 4 outcome for the RT sub-strand (roughly equating to the expected achievement of 12 to 13 year old students in Grades 6 and/or 7) reads: 'Students will recognise rotations, reflections and translations in arrangements and patterns, and translate, rotate and reflect figures and objects systematically'. At Levels 7 and 8 (equating to the expectations from Year 11/12 students),

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the RT outcome reads 'Students can identify the transformations needed to produce a given image from an original and applies transformations to problems including those involving congruent and similar shapes'. From these statements, the outcomes expectations from students are quite clear, and a range of assessment practices is now used by schools to gauge the attainment of these outcomes.

The topic of transformations lends itself conveniently to displaying the misperception phenomenon. The negative effects of misperceptions on learning have been largely unappreciated, and learning problems could have been misdiagnosed to be the result of other student difficulties. Recent cases involving students demonstrating various forms of mathematical misperceptions were found by Shaw, Durden and Baker (1998), Bottage (1999), and Malmer (2000). In one case, a student perceived a 90-degree angle correctly when aligned horizontally and vertically but perceived it as 140 degrees when tilted. A second case described a student misperceiving objects as reflections of these objects. In other scientific research, positrons were misperceived in data collected over many years, resulting in the non-identification of a new elementary particle. This misperception delayed an entire paradigm shift in quantum mechanics (Solomey, 1998).

Our own informal studies with school and university students and maths teachers (reported in this presentation) have revealed disturbing misperceptions involving reflections. Although these linear transformations are fundamental to maths, misperceptions involving them have not been studied in regular classrooms. There are many other situations in which we suspect that mathematics students are being hampered by misperception – for example, perceptual errors occur when students observe the vertex position while comparing two graphs (Goldenberg, 1988). Perceptual errors with respect to curved properties such as smoothness and concavity and slope were also displayed routinely by students in the above studies.

It is our contention that misperceptions seriously affect students' efforts to achieve a level of outcome appropriate for their age in this strand of the mathematics syllabus, and that the problem affects not only upper primary and lower secondary school students, but pre-service and practising teachers also. Teachers who perceive in their mind's eye something that is different to reality are likely to mis-teach. Students who misperceive will fail to see what the teacher is presenting (be it correct or incorrect) and hence cannot learn properly. The identification of these students and the diagnosis and remediation of their problems will result in a more efficient and effective learning environment. By producing a battery of crystallised examples of typical student misperceptions, teachers can be trained to identify them.

Another important aspect of the study is the interaction between the disciplines of mathematics and IT. The inter-disciplinary approach is in line with "several influential documents advocating the integration of mathematics, technology and other content areas" (Venville et al., 1999, p. 2). There are definite educational benefits in engaging student learning through integration, and some evidence of this is beginning to appear. This cross-pollination would bring new insights into the benefits of inter-disciplinary learning and professional collaboration.

Aims of the Presentation

To demonstrate that school and tertiary students, along with pre-service and practicing mathematics teachers experience misperceptions when working with linear transformations; to explore the effect of remediation via a software tool, and to examine some of the implications of the results.

Method

Participants

A sample of convenience was employed since there is no evidence to suggest that any particular group of students or teachers suffer more from learning or teaching problems caused by misperception. Six upper primary school and six lower secondary school students were drawn from primary and secondary schools in the Perth metropolitan area. Also, four undergraduate mathematics students along with four full-time pre-service mathematics teachers and four part-time practicing mathematics teachers enrolled in graduate level courses at a Perth university were also selected. There was no attempt on our part to seek out students who possessed a disposition to misperceive. Teachers in the primary and secondary schools were simply requested to allow students to participate in an activity with the software, and the tertiary participants were approached directly and requested to participate. From this sample of 24, four participants are described in this paper because they displayed the typical misperception phenomenon to a significant degree. Another seven participants displayed misperceptions to a lesser or trivial degree.

The Software

The instrument identified to possess the required functionality is *Mathemagic* (Lamb, 1995). By providing an interactive environment that is instantaneously responsive, this software is able to help us identify misperceptions demonstrated in the workings of individual learners. The *Mathemagic* software allows students to explore the spatial transformations of reflection, enlargement, translation and rotation. The transformations can be applied individually or in combination, and the software provides opportunities for problem-solving using the following operations:

(a) Translation (sliding) up or down by one unit; left or right by one unit.
(b) Rotation through a quarter turn (90°) clockwise or anti-clockwise.
(c) Dilatation (scaling) by a factor of 2 or 1/2.
(d) Reflection in the x- or y-axis, or in the line $y = x$, or in the line $y = -x$.

A typical transformation task set for problem-solving is presented as follows: the original picture on the screen (in black), is shown together with the end result (in blue) of several transformations. The challenge for the student is to find a combination of operations that turns the (original) black picture into the (transformed) blue one. The solution may be entered in one of two ways: (1) each of the student's moves may be shown when it is selected, or (2) the moves may only be viewed *after* all the moves have been chosen, thus requiring the student to visualize the entire sequence of transformations. The software has a feature that challenges the student in visualising the least number of correct moves possible.

The Approach

In contrast to most other subjects where the computer can only simulate reality, this software permits us to work with real mathematics. Virtual reality is a pale substitute for the real thing. Furthermore, virtual reality is perceived by many students as 'fictitious', thus it is not always believable to them (Yeo, Loss, Treagust & Zadnick 1999; Gorsky & Finegold, 1994). Because technology provides the explicit reality corresponding to mathematical models, students can directly perceive the validity of the models which they manipulate on the computer screen and thereby make the essential connections to effect authentic learning. In this portion of our study, the *Mathemagic* software has been used to diagnose several cases of misperceptions. Participants were first allowed time to become familiar with this software, manipulating a bird object (Figure 1 – used in the problem presented to them later) according to the four transformation operations listed above. A bird object was chosen to manipulate because of its lack of symmetry when viewed from the side. Any shape could be used and is easily created with the software.

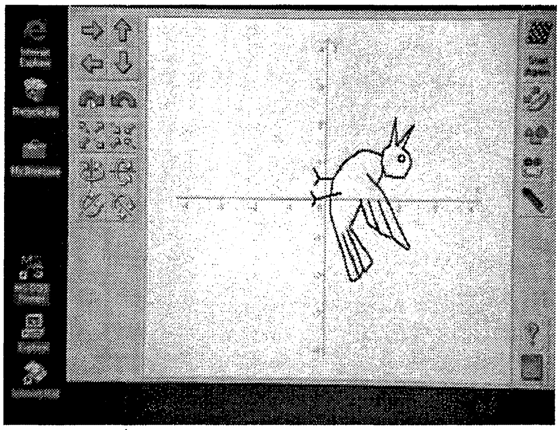

Figure 1. The bird object.

The steps in the procedure used with each student were as follows: After 30 minutes practice with the *Mathemagic* software operations, the participants watched as a sketch of the bird was drawn on paper. They were then told a transformation to apply to the sketch (the reality), mentally carry out the steps involved in the transformation (the imagined reality), and then sketch the result on the same piece of paper. Any discrepancy could be due to faulty visualisation or faulty drawing skills, so the significant parts of the picture and its transformation were examined to deduce the occurrence of misperceptions. The correct orientation of the bird was shown to the participant. A record was kept of the participants' comments and drawing attempts, and their reactions on being presented with the computer-generated solution were also encouraged and recorded. We were interested in seeing if the participants were able to predict the transformation that they we knew they understood both conceptually and cognitively. The task confronted the participants' powers of visualisation or perception, with the program providing the reality of the task.

Examples of Misperceptions

Reproduced below are the drawing efforts of four of the participants Each example is accompanied by: (a) details of the task, (b) details of the participant, and (c) their reactions on being presented with the correct transformation.

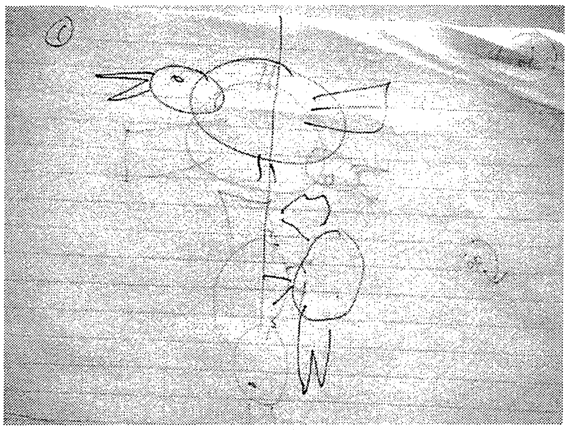

Figure 2(a). Student A.

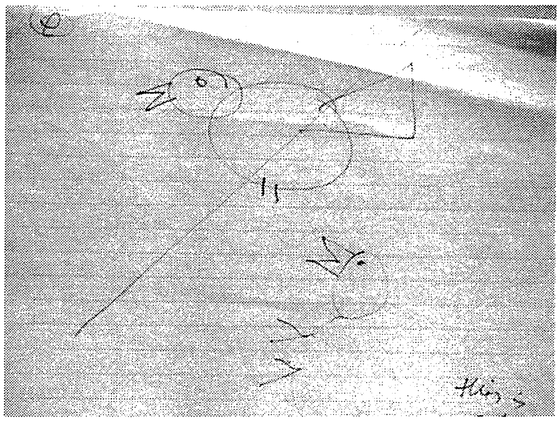

Figure 2(b). Student A's re-try.

Figures 2(a) and (b) are the work of Student A, a male teacher of mathematics and science studying full time for a postgraduate degree at a Perth University. These were A's second and third attempts at this procedure – he had gone through the sketching activity two weeks earlier (without much success). In Figure 2(a), the bird object was sketched at the top of the page and A was requested to reflect the bird about the vertical axis that was drawn in on the figure. After a few moments thought, A sketched the bird as being rotated through 90°, and then reflected about the y-axis. He then used the software to perform the requested translation and was surprised at the outcome. In order to check if A had misunderstood what he had been asked to do, he was then asked if he would like to re-try with a second example. In Fig. 2(b) the uppermost sketch was drawn and A was asked to sketch the bird if it was reflected in the line $y = x$.

Again after several minutes thought, A sketched the figure in the lower half of the Figure which represented a rotation through 90° followed by a translation to the right. On using the software to carry out the transformation requested, A was incredulous at the result. He spent some time afterwards experimenting with the software, striving to perform each transformation again. He was eventually successful with both after some 10 minutes – indicative of the value of practice – but he continued to experience trouble dealing with the actual and the imagined reality of the activity.

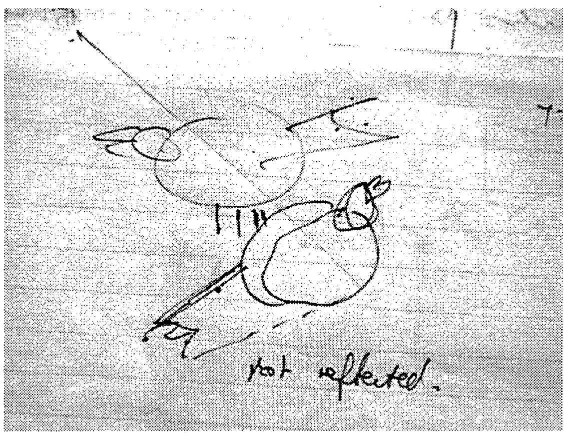

Figure 3. Student B.

Figure 3 represents the work of Student B, a male pre-service teacher of computing. He was asked to reflect the bird at the top of Figure 3 about the line $y = -x$, drawn on the diagram. B sketched the figure in the lower portion of Fig 3 after only a few moments thought. It represents a rotation through 180° about the line y = –x (correct), followed by a reflection in the x-axis (incorrect). He then used the software to reproduce his thought sequence. He recognised the requested transformation after the first translation when he used the correct software operation. B was annoyed with himself, saying that he could not understand why he had gone on to perform the additional incorrect operation. He said that the task became clearer when he had the options of the various computer operations to follow. Student B had no wish to spend more time gaining experience with similar activities, saying that he was confident that with practice he would be able to deal with any task given to him.

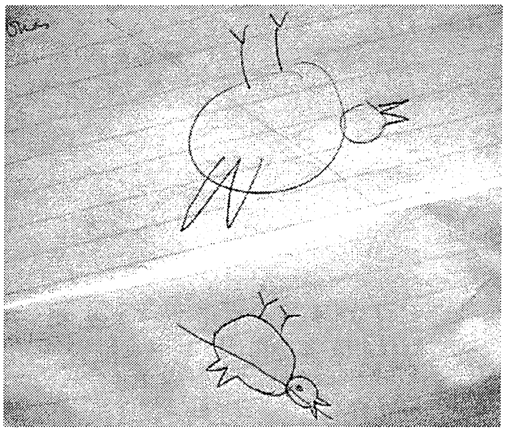

Figure 4. Student C.

Figure 4 represents the work of Student C, a Grade 7 primary school female. After checking that she knew the meaning of the term 'rotate' and could used the software correctly to carry out this transformation, C was asked to rotate the bird through the line y = –x drawn on the bird in the uppermost section of Figure 4.

Her response, carried out after several minutes consideration, represents a rotation through 45° in the plane of the line y = –x. On using the software to carry out the requested exercise C exclaimed 'of course! – It's nothing like it, is it?'' Student C described to us what should have occurred – pointing out where the correct position of the tail feathers should have been, and why. She practised with other positions of the bird, checking with the software each time for a considerable period afterwards and became very adept at all manner of transformations before she terminated the activity. At one point student C made the comment: "it's good to see the right picture at the end. It's like checking your answers from the back of the book". Practice coupled with the conflict situation had clarified the difference between the actual and the imagined reality for C.

It is worth recording that student C was the single female to be included among the four participants demonstrating the most pronounced examples of misperceptions. The original sample for this component of the larger study consisted of twenty-four students – the four described herein, and another seven demonstrating misperceptions to a lesser degree. Student C was the only female among this group of eleven participants. We propose to investigate the typicality of this situation in the larger study.

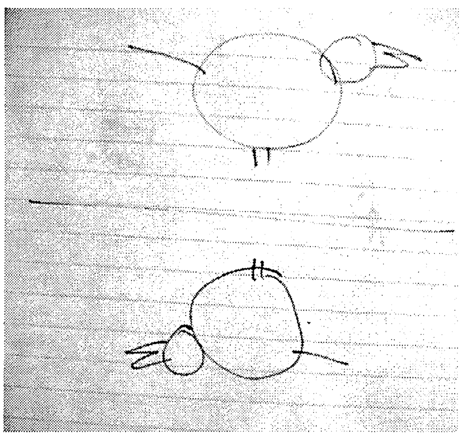

Figure 5. Student D.

Figure 5 depicts the effort of Student D, a male first year university student enrolled in a BSc (Mathematics) program. He was asked to reflect the bird sketched in the uppermost section of the Figure about the x-axis. It can be seen from his sketch in the bottom half of the Figure that he first reflected the bird about the x-axis as required, but then rotated the bird through 180°. On using the software to carry out the requested transformation he quickly recognised the error he had made, but could not immediately decide what he had done incorrectly when imagining /visualising the steps that he had to take. "I imagined that the reflection would also turn the bird" he said. "I don't know why". Student D asked to spend further time practicing and then checking his efforts against the software, and after almost one hour, was performing a variety of transformations confidently and successfully.

Discussion

The four illustrations described in this paper suggest that the outcomes of the larger study could have far-reaching implications for educational practice. The existence of the misperception phenomenon in the teaching and learning of linear transformations cannot be disputed, and it is probable that it would have an effect in other subject areas too. It is of some concern that both teacher and learner may experience the same problem, but because of the lack of attention given to this phenomenon in the past, the problem may have been attributed to other forms of teaching/learning difficulty or even passed unnoticed.

It was interesting that more male than female participants from our sample seemed to experience misperceptions in linear transformation tasks. This seemed to be also true when considering the age groups represented. However, more research would be needed to verify this possibility that there is a gender factor in the misperception phenomenon.

While some of the participants had the opportunity to spend time on the software for remedial purposes, most of them were unable to devote the time for this activity. It is gratifying that all of those who took part in remedial activities improved in carrying out the correct transformations.

We hope to produce a battery of examples of typical student misperceptions, the existence of which will assist teachers to better understand student learning difficulties resulting from misperceptions and facilitate the training of teachers to identify misperceptions when they occur.

The development and customisation of diagnostic and remediation software tools of the type used here has the potential to apply our approach to other mathematical topics. Such development will also facilitate the testing of large groups of students, enabling those who do experience misperception to be identified early and be assisted in overcoming some of the negative outcomes of the phenomenon. The computer-aided approach enabled the participants described in this paper to be presented visually with the discrepancies between their imagined reality of the transformations requested of them and the actual reality of the transformation that the software presented to them. This was a real 'conflict teaching' situation that, particularly in the case of Student A, forcefully brought home to the students the extent of their misperceptions. We are hopeful that the outcomes of our research will assist in overcoming the problems associated with the misperception phenomenon and consequently contribute to an improvement in the quality of teaching and learning in mathematics at all levels.

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